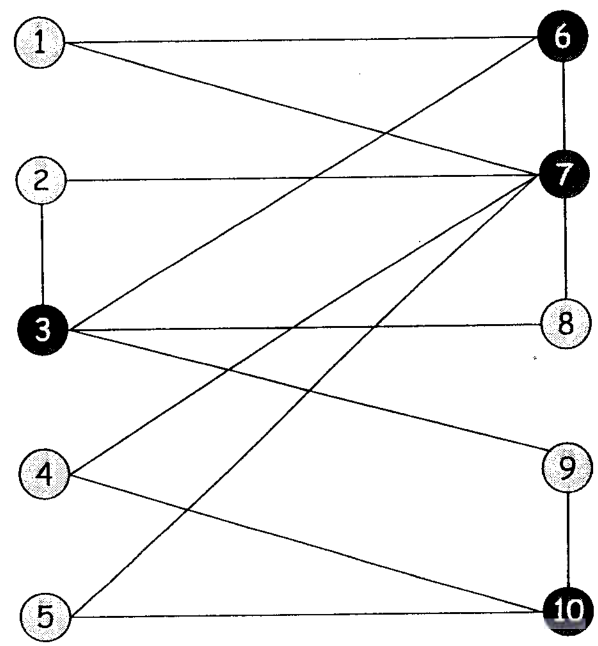
Vertex Cover

VERTEX COVER: *Given a graph G* = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$, or $v \in S$, or both.



k = 4
S = { 3, 6, 7, 10 }

Finding Small Vertex Covers

Q. What if k is small?

Brute force. $O(k\, n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\, n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k, e.g., to $O(2^k\, k\, n)$.

Ex. $n = 1{,}000$, $k = 10$.
Brute. $k\, n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k\, k\, n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, algorithm is poly-time; if k is a small constant, then it's also practical.

Finding Small Vertex Covers

Claim. Let u-v be an edge of G. G has a vertex cover of size $\leq k$ iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\leq k-1$.

delete v and all incident edges

Pf. $\Rightarrow$

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

Pf. $\Leftarrow$

- Suppose S is a vertex cover of $G - \{u\}$ of size $\leq k-1$.
- Then $S \cup \{u\}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\leq k(n-1)$ edges.
Pf. Each vertex covers at most n-1 edges. ▪

Finding Small Vertex Covers: Algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\le k$ in $O(2^k kn)$ time.

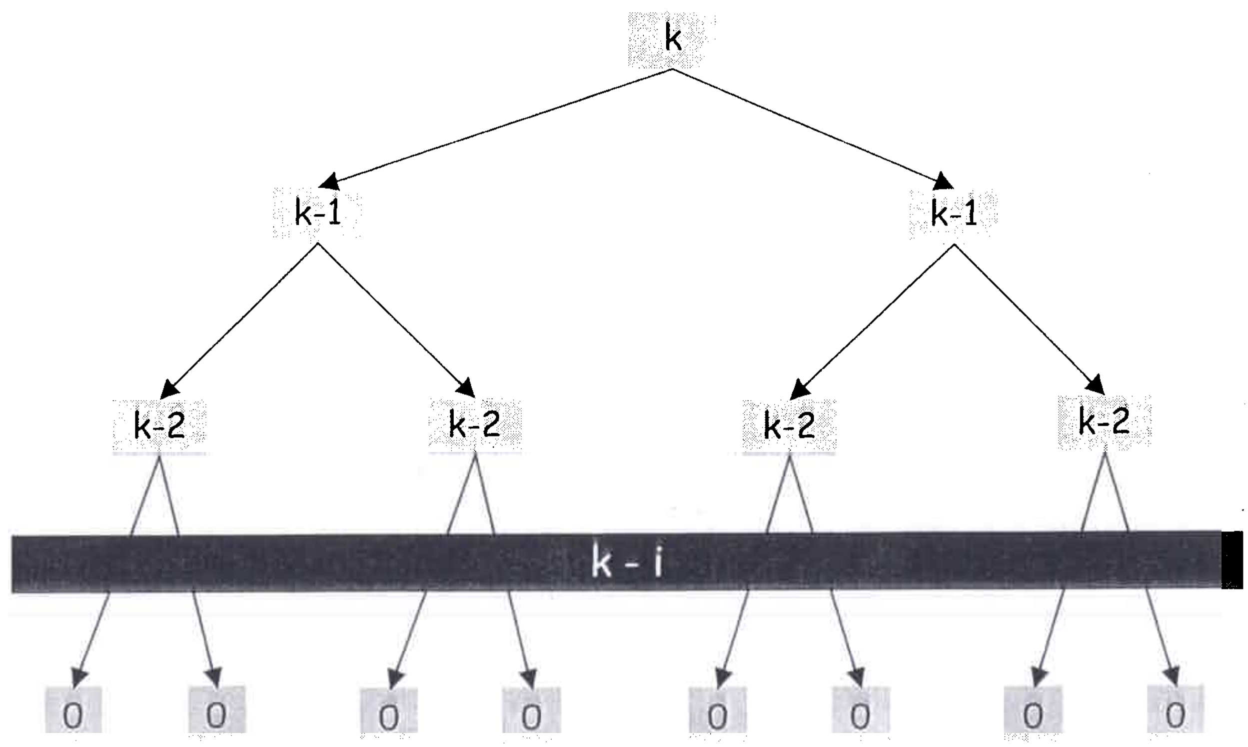
```
boolean Vertex-Cover(G, k) {
   if (G contains no edges)    return true
   if (G contains ≥ kn edges) return false

   let (u, v) be any edge of G
   a = Vertex-Cover(G - {u}, k-1)
   b = Vertex-Cover(G - {v}, k-1)
   return a or b
}
```

Pf.

- Correctness follows previous two claims.
- There are $\le 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. ▪

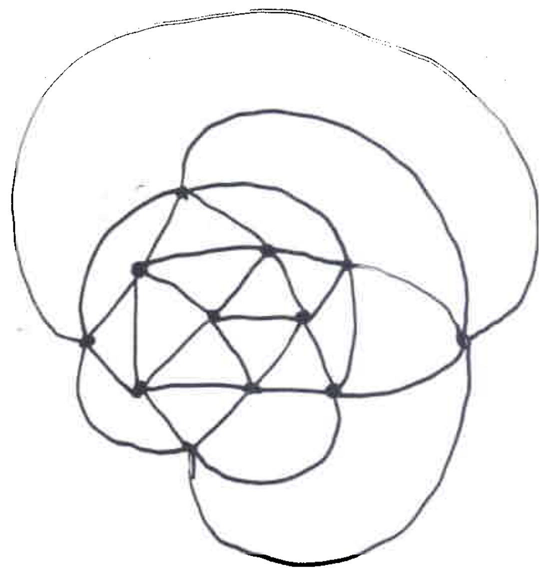
Finding Small Vertex Covers: Recursion Tree

$$T(n,k) \le \begin{cases} cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k c k n$$



Planar Graphs, Four-Coloring, and Hamiltonian Cycles

Duality: Vertices ↔ Faces



Planar Graphs, Four-Coloring, and Hamiltonian Cycles

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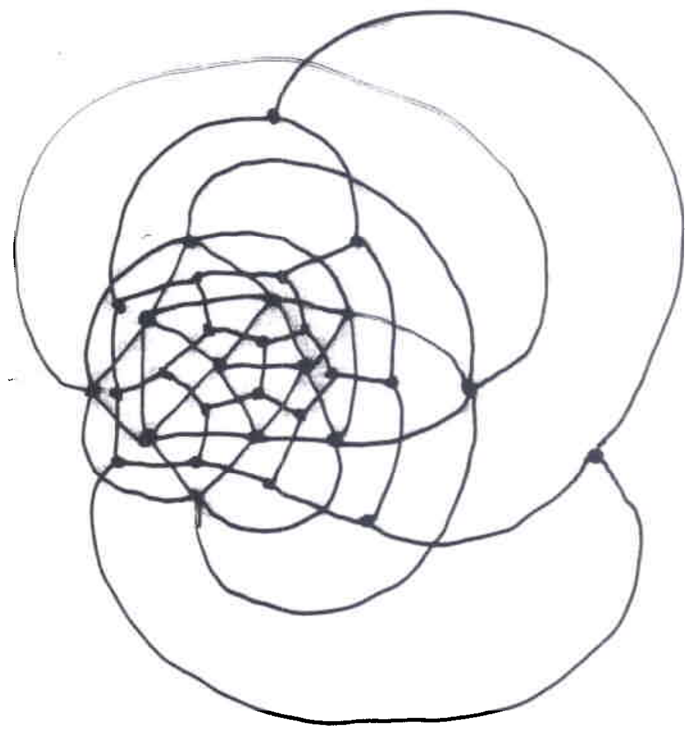
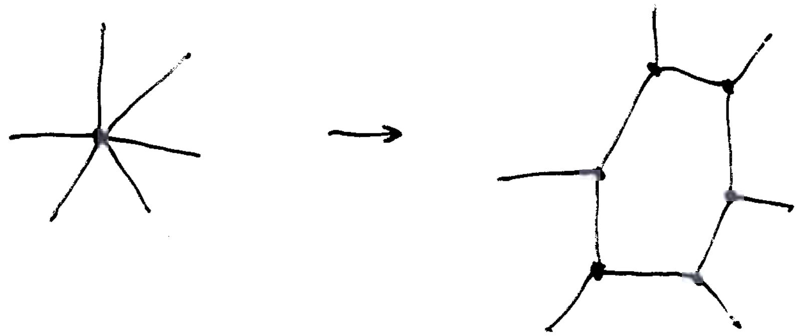


Vertex-coloring ↔ Face coloring

Reduce to degree 3:



Hamiltonian cycle → 4 coloring

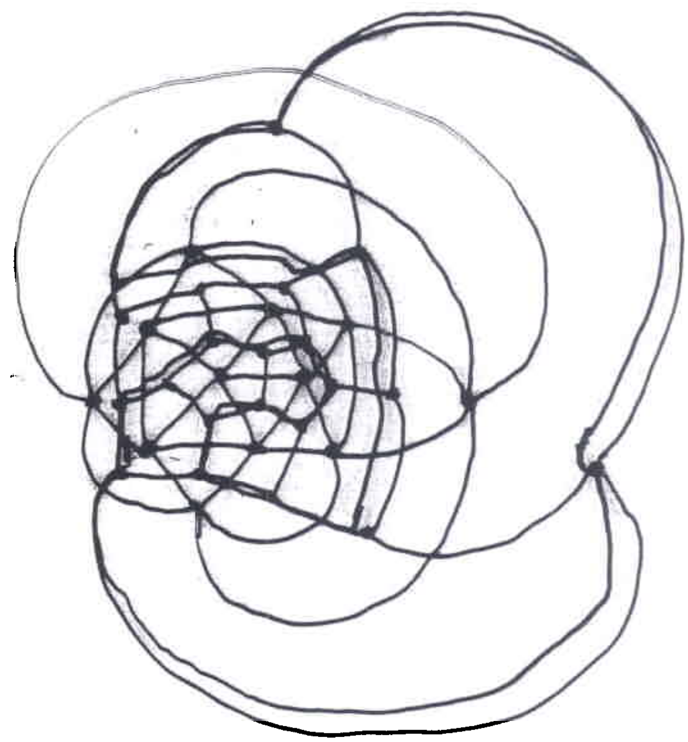
Color inside with 2 colors,

outside with 2 colors:

each side has no cycles

Planar Graphs, Four-Coloring, and Hamiltonian Cycles

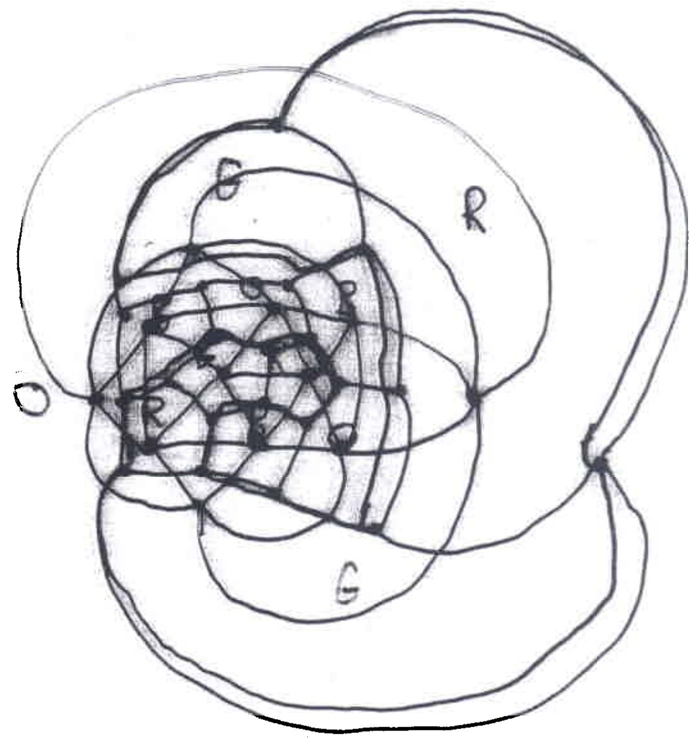
Duality: Vertices ↔ Faces



Vertex-coloring ↔ Face coloring

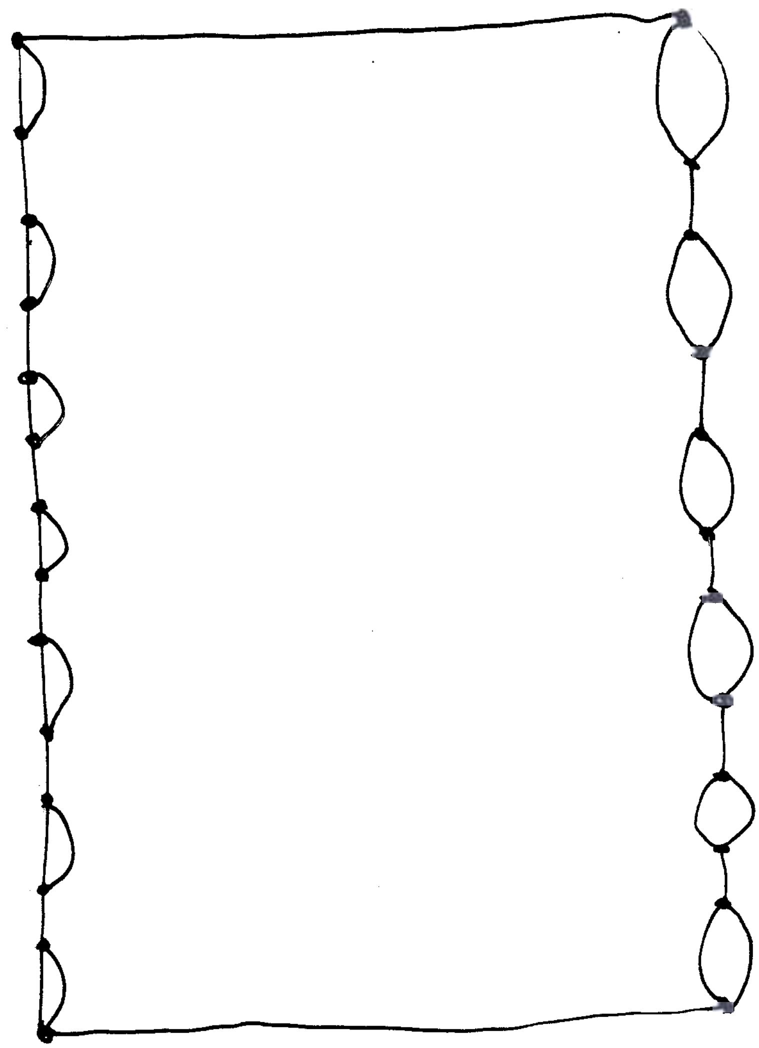
Planar Graphs, Four-Coloring, and Hamiltonian Cycles

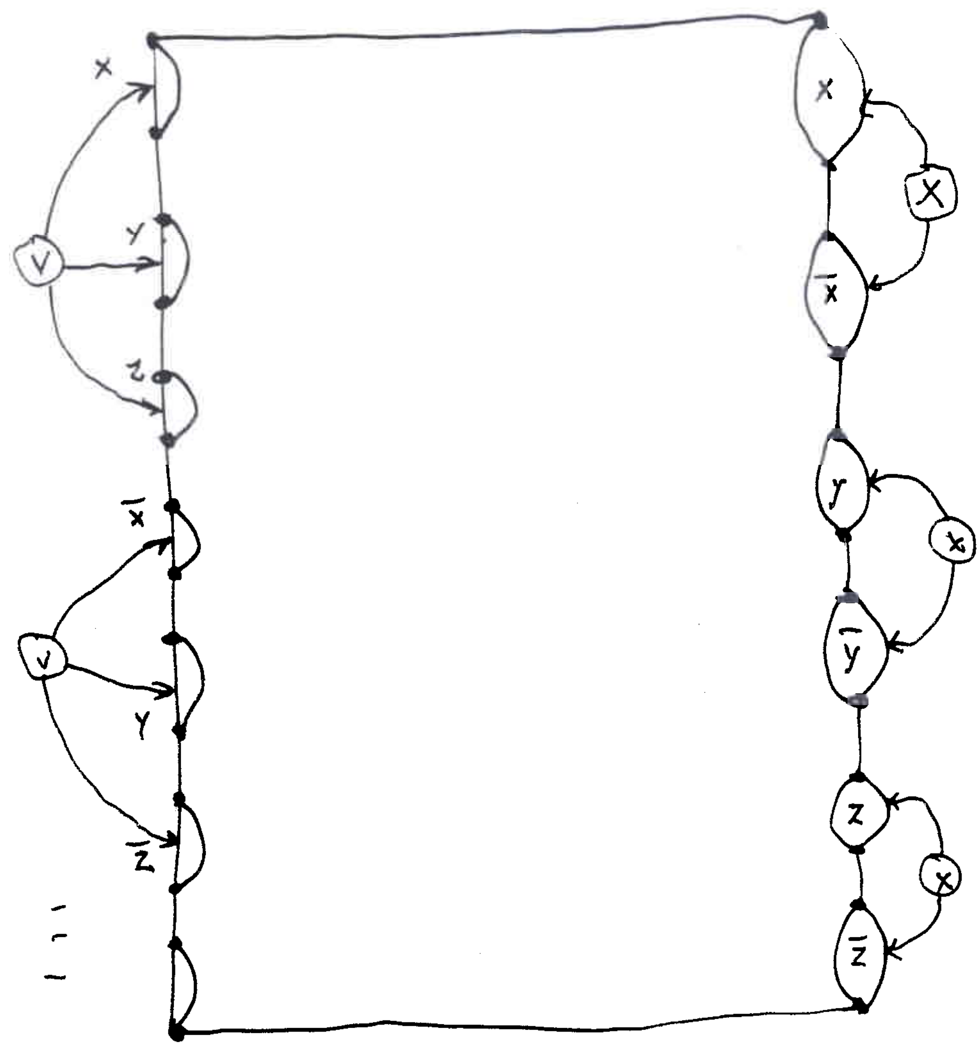
Duality: Vertices ↔ Faces

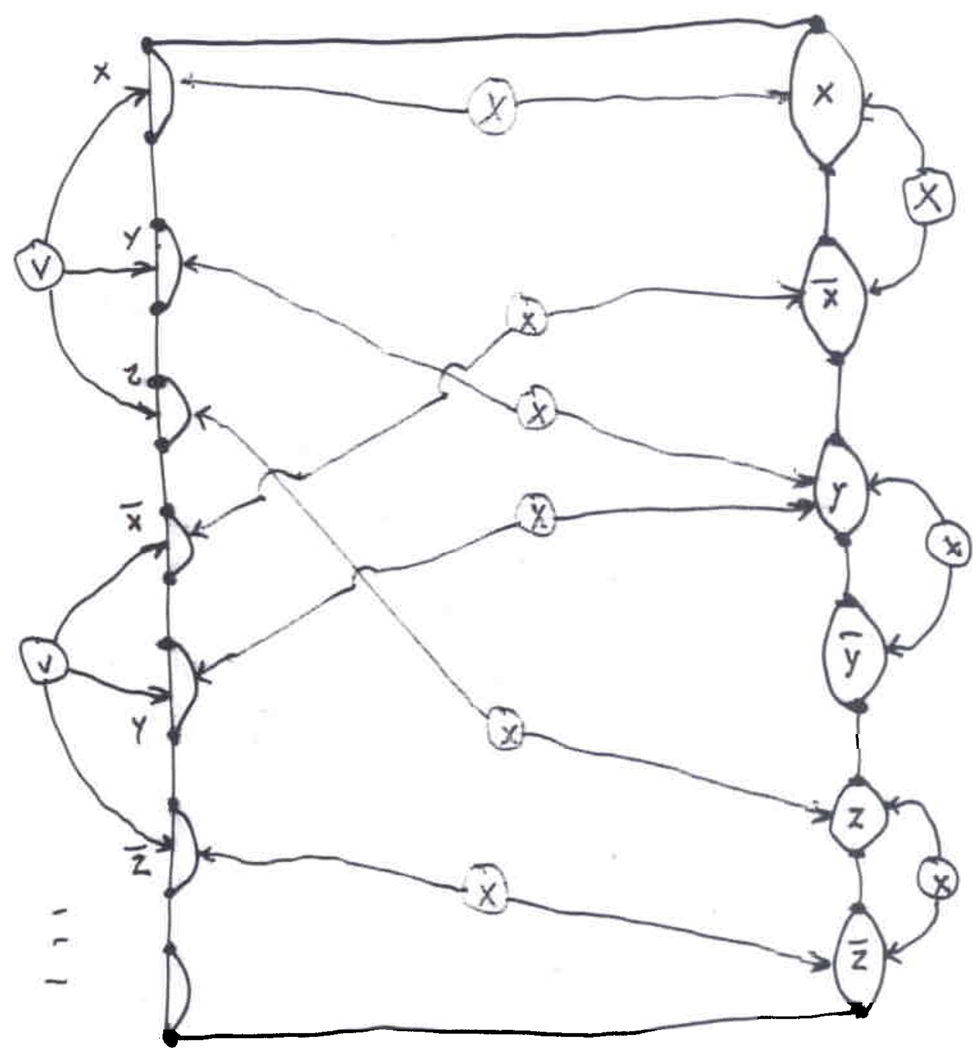


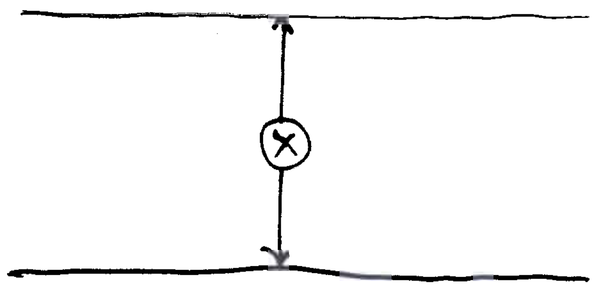
Vertex-coloring ↔ Face coloring

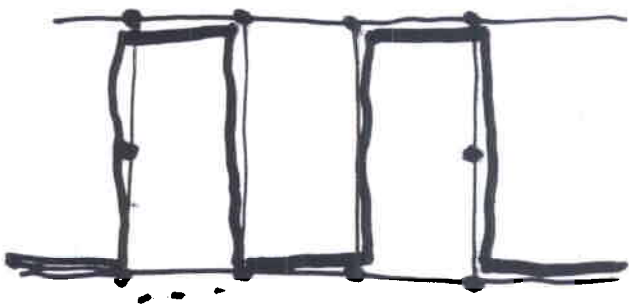
NP-completeness of Planar Hamilton Cycle

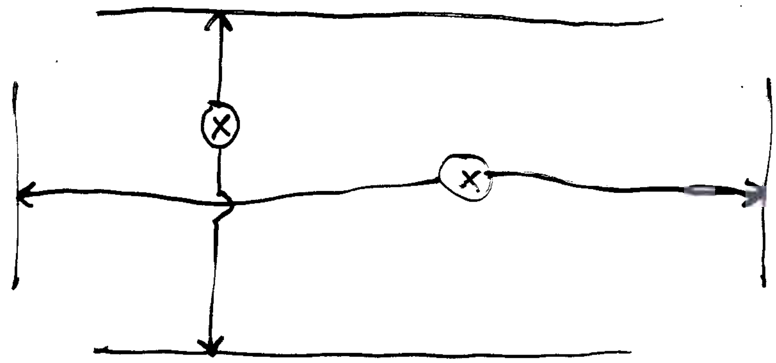


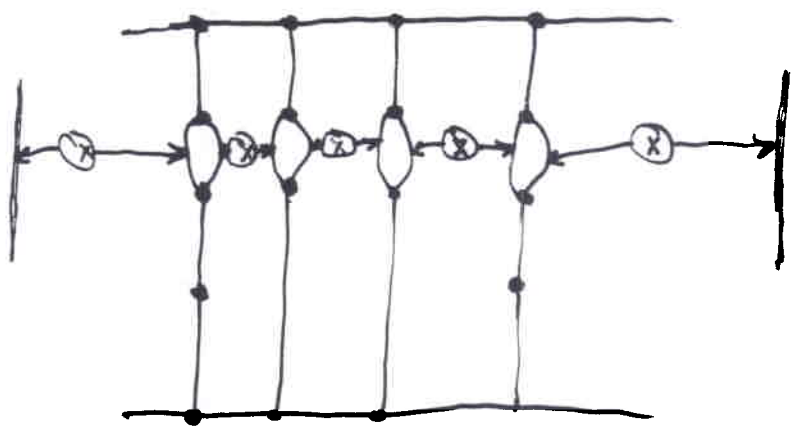









x
x


x
x
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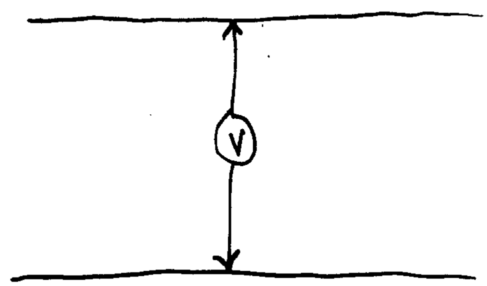
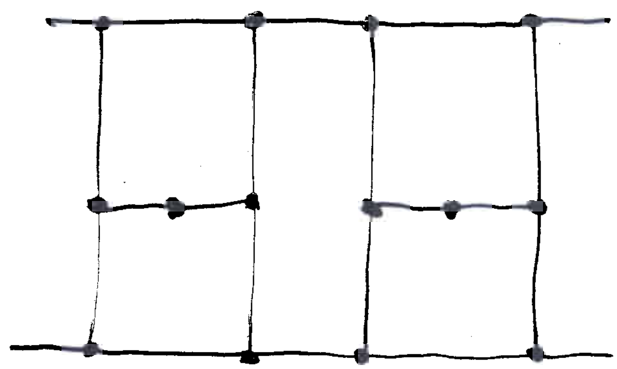
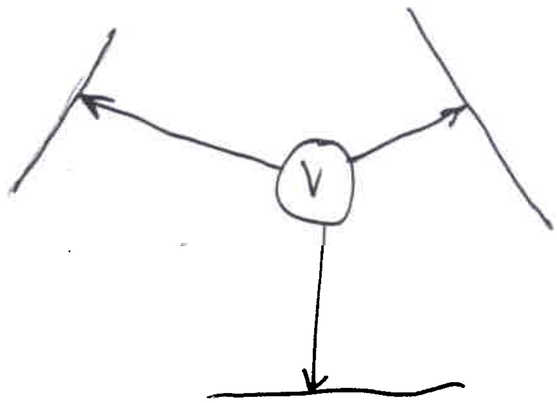
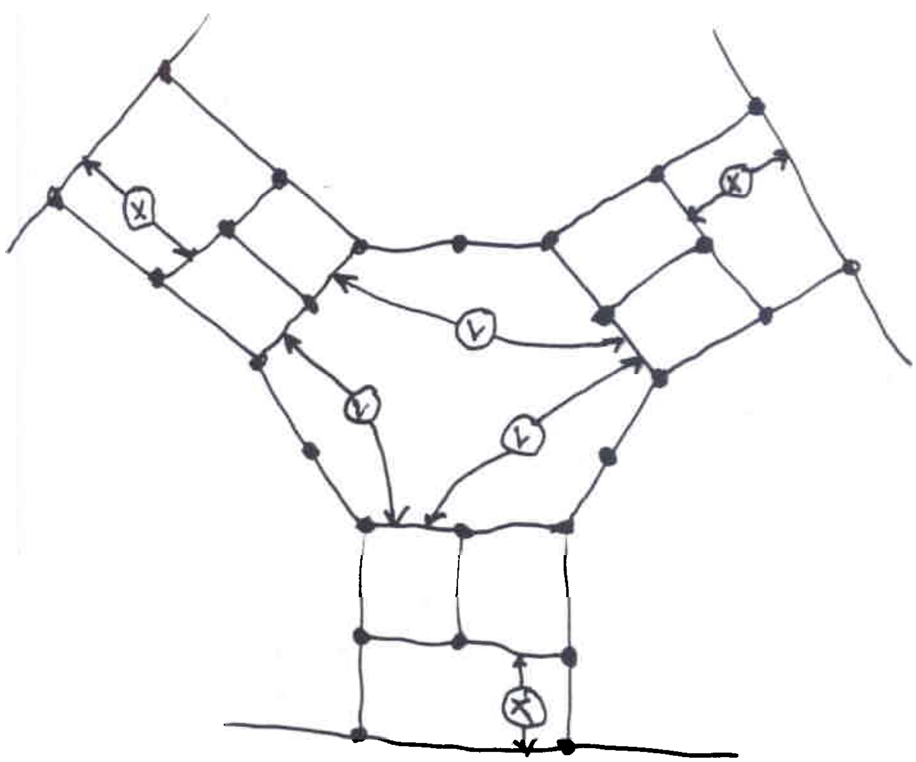

V




V


X
X
V
V
V
X